Necessary and Sufficient Conditions for the Exitsence of Metric in Three-Dimensional .Affine Manifolds*

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Explicit necessary and sufficient conditions for the existence of metric in ***three-dimensional*** affine manifolds are found in this paper. These conditions can be grouped into two kinds: (i) those involving the covariant derivatives of the Riemannian tensor $R^{\alpha}{}_{\beta\gamma\delta}$, and (ii) those involving $R^{\alpha}{}_{\beta\gamma\delta}$ ***only.*** The first group consists of eighteen equations of the form $(R^2{}_{131}R^3{}_{112}-R^3{}_{131}R^2{}_{112})(R^1{}_{131}R^3{}_{112}-R^3{}_{131}R^1{}_{112})_{;\lambda}=(R^2{}_{131}R^3{}_{112}-R^3{}_{131}R^2{}_{112})_{;\lambda}\cdot(R^1{}_{131}R^3{}_{112}-R^3{}_{131}R^1{}_{112})$ and their permutations. The second group contains three third-degree conditions, three fourth-degree conditions and thirteen sixth-degree conditions. In case the above necessary and sufficient conditions are satisfied, the solutions for metric are obtained.

I. INTRODUCTION

WITH the motivation (i) of intrinsic mathematical importance and (ii) of many applications to theoretical physics, especially gravitational physics, we started to embark on the project of obtaining the necessary and sufficient conditions for affine manifolds to have a metric in a recent paper[1]. In that paper, we solved the problem for the two-dimensional case. In fact, we proved the following theorem:

THEOREM: The necessary and sufficient confitions for the existence of a (local) metric in two-dimensional affine manifolds are

(i) $R_{12}=R_{21}$,

(ii) $R^1{}_{112}R^2{}_{112;1}=R^2{}_{112}R^1{}_{112;1}$, $\quad R^1{}_{112}R^2{}_{112;2}=R^2{}_{112}R^1{}_{112;2}$,
$R^1{}_{112}R^1{}_{212;1}=R^1{}_{212}R^1{}_{112;1}$, $\quad R^1{}_{112}R^1{}_{212;2}=R^1{}_{212}R^1{}_{112;2}$.

In this paper we find the necessary and sufficient conditions for the three dimensional case. The second group (ii) of conditions above remains similar. The first group (i) of condition becomes a group consisting of three third-degree conditions, three fourth-degree conditions and thirteen sixth-degree conditions in $R^{\alpha}{}_{\beta\gamma\delta}$. In Section II, we obtain some necessary conditions. In Section III, we find the necessary and sufficient conditions and give the solutions when these conditions are satisfied.

II. SOME NECESSARY CONDITIONS

Let M be an N-dimensional affine manifold with (symmetric) connection $\Gamma^{\alpha}{}_{\beta\gamma}$ and Riemannian tensor

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(1) K. S. Cheng, and W. T. Ni, Chinese J. of Phys. 16, No. 4, 223 (1978).

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} \tag{1}$$

We will, in this section, derive some necessary conditions for a (nonsingular) metric $g_{\alpha\beta}$ to exist such that the affine connection can be obtained from this metric as in a Riemannian manifold, i.e.,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}). \tag{2}$$

We first consider general N and then specialize to $N=3$. Throughout this paper, we use the convention that "," denotes partial differentiation and ";" denotes covariant differentiation.

Equations (2) are equivalent to

$$g_{\alpha\beta,\gamma} = \Gamma^{\mu}{}_{\alpha\gamma} g_{\mu\beta} + \Gamma^{\mu}{}_{\beta\gamma} g_{\alpha\mu} \tag{3}$$

or,

$$g_{\alpha\beta;\gamma} = 0. \tag{4}$$

Using the covariant-differentiation-order-change relation

$$S_{\alpha\beta;\gamma\delta} = S_{\alpha\beta;\delta\gamma} + R^{\mu}{}_{\alpha\gamma\delta} S_{\mu\beta} + R^{\mu}{}_{\beta\gamma\delta} S_{\alpha\mu} \tag{5}$$

for an arbitrary second rank tensor $S_{\alpha\beta}$ and equations (4), we readily obtain

$$g_{\mu\beta} R^{\mu}{}_{\alpha\delta\gamma} + g_{\alpha\mu} R^{\mu}{}_{\beta\delta\gamma} = 0. \tag{6}$$

Covariant-differentiating (6) and using (4), we obtain

$$g_{\mu\beta} R^{\mu}{}_{\alpha\delta\gamma;\epsilon} + g_{\alpha\mu} R^{\mu}{}_{\beta\delta\gamma;\epsilon} = 0, \tag{7}$$

$$g_{\mu\beta} R^{\mu}{}_{\alpha\delta\gamma;\epsilon\lambda} + g_{\alpha\mu} R^{\mu}{}_{\beta\delta\gamma;\epsilon\lambda} = 0, \tag{8}$$

etc.

Multipllyng (6) by $g^{\alpha\beta}$ and summing over β, we derive

$$R^{\alpha}{}_{\alpha\delta\gamma} = 0, \tag{9}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\mu\nu}$ such that

$$g^{\alpha\beta} g_{\beta\mu} = \delta^{\alpha}{}_{\mu}. \tag{10}$$

For an affine manifold the conditions (9) are equivalent to

$$R_{\alpha\beta} = R_{\beta\alpha} \tag{11}$$

due to the identity $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$ where $R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\gamma\beta}$ is the Riccl tensor.

Now we specialize to $N=3$. Let $(\alpha, \beta, \gamma, \delta)$ equals to (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 2, 3) respectively in (6), we have

$$\begin{aligned} g_{11} R^1{}_{112} + g_{12} R^2{}_{112} + g_{13} R^3{}_{112} &= 0, \\ g_{11} R^1{}_{113} + g_{12} R^2{}_{113} + g_{13} R^3{}_{113} &= 0, \\ g_{11} R^1{}_{123} + g_{12} R^2{}_{123} + g_{13} R^3{}_{123} &= 0. \end{aligned} \tag{12}$$

For (12) to have a nontrivial solution of (g_{11}, g_{12}, g_{13}) the following determinant must vanish:

$$\det\begin{pmatrix} R^1{}_{112} & R^2{}_{112} & R^3{}_{112} \\ R^1{}_{113} & R^2{}_{113} & R^3{}_{113} \\ R^1{}_{123} & R^2{}_{123} & R^3{}_{123} \end{pmatrix} = 0. \tag{13}$$

When (13) hold the ratios of g's are as follows:

$$\frac{g_{12}}{g_{11}} = -\frac{R^1{}_{112} R^3{}_{113} - R^1{}_{113} R^3{}_{112}}{R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112}} \tag{14}$$

$$\frac{g_{13}}{g_{11}} = -\frac{R^1{}_{113} R^2{}_{112} - R^1{}_{112} R^2{}_{113}}{R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112}} \tag{15}$$

Similarly since

$$g_{\alpha 1} R^{1}{}_{\alpha\gamma\delta}+g_{\alpha 2} R^{2}{}_{\alpha\gamma\delta}+g_{\alpha 3} R^{3}{}_{\alpha\gamma\delta}=0, \tag{16}$$

$$g_{\alpha 1} R^{1}{}_{\alpha\bar{\gamma}\bar{\delta}}+g_{\alpha 2} R^{2}{}_{\alpha\bar{\gamma}\bar{\delta}}+g_{\alpha 3} R^{3}{}_{\alpha\bar{\gamma}\bar{\delta}}=0, \tag{17}$$

$$g_{\alpha 1} R^{1}{}_{\alpha\bar{\bar{\gamma}}\bar{\bar{\delta}}}+g_{\alpha 2} R^{2}{}_{\alpha\bar{\bar{\gamma}}\bar{\bar{\delta}}}+g_{\alpha 3} R^{3}{}_{\alpha\bar{\bar{\gamma}}\bar{\bar{\delta}}}=0. \tag{18}$$

(no summation over the index a)

we have

$$\det\begin{pmatrix} R^{1}{}_{\alpha\gamma\delta} & R^{2}{}_{\alpha\gamma\delta} & R^{3}{}_{\alpha\gamma\delta} \\ R^{1}{}_{\alpha\bar{\gamma}\bar{\delta}} & R^{2}{}_{\alpha\bar{\gamma}\bar{\delta}} & R^{3}{}_{\alpha\bar{\gamma}\bar{\delta}} \\ R^{1}{}_{\alpha\bar{\bar{\gamma}}\bar{\bar{\delta}}} & R^{2}{}_{\alpha\bar{\bar{\gamma}}\bar{\bar{\delta}}} & R^{3}{}_{\alpha\bar{\bar{\gamma}}\bar{\bar{\delta}}} \end{pmatrix}=0 \tag{19}$$

we can also use equation (7) or (8) to replace some of the equations (16)–(18) to obtain conditions that involve covariant derivatives of $R^{\alpha}{}_{\beta\gamma\delta}$.

III. NECESSARY AND SUFFICIENT CONDITIONS

In this Section we look into integrability conditions of equations (3) for three-dimensional case. Written more explicitly, equations (3) are

$$g_{11,\gamma}=2\Gamma^{1}{}_{1\gamma}\, g_{11}+2\Gamma^{2}{}_{1\gamma}\, g_{12}+2\Gamma^{3}{}_{1\gamma}\, g_{13}, \tag{20}$$

$$g_{22,\gamma}=2\Gamma^{1}{}_{2\gamma}\, g_{12}+2\Gamma^{2}{}_{2\gamma}\, g_{22}+2\Gamma^{3}{}_{2\gamma}\, g_{23}, \tag{21}$$

$$g_{33,\gamma}=2\Gamma^{1}{}_{3\gamma}\, g_{13}+2\Gamma^{2}{}_{3\gamma}\, g_{23}+2\Gamma^{3}{}_{3\gamma}\, g_{33}, \tag{22}$$

$$g_{12,\gamma}=(\Gamma^{1}{}_{1\gamma}+\Gamma^{2}{}_{2\gamma})g_{12}+\Gamma^{1}{}_{2\gamma}\, g_{11}+\Gamma^{2}{}_{1\gamma}\, g_{22}+\Gamma^{3}{}_{2\gamma}\, g_{13}+\Gamma^{3}{}_{1\gamma}\, g_{23}, \tag{23}$$

$$g_{13,\gamma}=(\Gamma^{1}{}_{1\gamma}+\Gamma^{3}{}_{3\gamma})g_{13}+\Gamma^{1}{}_{3\gamma}\, g_{11}+\Gamma^{3}{}_{1\gamma}\, g_{33}+\Gamma^{2}{}_{3\gamma}\, g_{12}+\Gamma^{2}{}_{1\gamma}\, g_{23}, \tag{24}$$

$$g_{23,\gamma}=(\Gamma^{2}{}_{2\gamma}+\Gamma^{3}{}_{3\gamma})g_{23}+\Gamma^{2}{}_{3\gamma}\, g_{22}+\Gamma^{3}{}_{2\gamma}\, g_{33}+\Gamma^{1}{}_{3\gamma}\, g_{12}+\Gamma^{1}{}_{2\gamma}\, g_{13}. \tag{25}$$

For convenience, we define

$$(a_{ij})\equiv\begin{pmatrix} R^{1}{}_{123} & R^{2}{}_{123} & R^{3}{}_{123} \\ R^{1}{}_{131} & R^{2}{}_{131} & R^{3}{}_{131} \\ R^{1}{}_{112} & R^{2}{}_{112} & R^{3}{}_{112} \end{pmatrix} \tag{26}$$

From equations (13)–(15), we have

$$\det(a_{ij})=0, \tag{27}$$

$$g_{12}= A^{31}\quad g_{11}, \tag{28}$$

$$g_{13}=\frac{A^{31}}{A^{11}}\, g_{11}, \tag{29}$$

where A^{ij} is the cofactor of a_{ji} in the matrix (26). Substituting (28), (29) into (20), we get

$$g_{11,\lambda}=2g_{11}\left[\Gamma^{1}{}_{1\lambda}+\Gamma^{2}{}_{1\lambda}\,\frac{A^{21}}{A^{11}}+\Gamma^{3}{}_{1\lambda}\,\frac{A^{31}}{A^{11}}\right]. \tag{30}$$

Dividing by $2g_{11}$ and differentiating, we have

$$\left(\frac{1}{2}\ln|g_{11}|\right)_{,\lambda\delta}=\Gamma^{1}{}_{1\lambda,\delta}+\Gamma^{2}{}_{1\lambda,\delta}\,\frac{A^{21}}{A^{11}}+\Gamma^{3}{}_{1\lambda,\delta}\,\frac{A^{31}}{A^{11}}$$
$$+\Gamma^{2}{}_{1\lambda}\left(\frac{A^{21}}{A^{11}}\right)_{,\delta}+\Gamma^{3}{}_{1\lambda}\left(\frac{A^{31}}{A^{11}}\right)_{,\delta}. \tag{31}$$

Now the intcgrability conditions $(\frac{1}{2}\ln|g_{11}|)_{,\lambda\delta}-(\frac{1}{2}\ln|g_{11}|)_{,\delta\lambda}=0$ become

$$(\Gamma^{1}{}_{1\lambda,}\,a-\Gamma^{1}{}_{1\lambda,\delta})+(\Gamma^{2}{}_{1\lambda,\delta}-\Gamma^{2}{}_{1\delta,\lambda})\,\frac{A^{21}}{A^{11}}+(\Gamma^{3}{}_{1\lambda,\delta}-\Gamma^{3}{}_{1\delta,\lambda})\,\frac{A^{31}}{A^{11}}$$
$$+\Gamma^{2}{}_{1\lambda}\left(\frac{A^{21}}{A^{11}}\right)_{,\delta}-\Gamma^{2}{}_{1\delta}\left(\frac{A^{21}}{A^{11}}\right)_{,\lambda}+\Gamma^{3}{}_{1\lambda}\left(\frac{A^{31}}{A^{11}}\right)_{,\delta}-\Gamma^{3}{}_{1\delta}\left(\frac{A^{31}}{A^{11}}\right)_{,\lambda}=0. \tag{32}$$

Multiplying (32) by $(A^{11})^2$ and using (1) to convert the derivatives of Γ's into other quantities, we obtain

$$
\begin{aligned}
&(R^1{}_{1\delta\lambda}-\Gamma^\beta{}_{1\lambda}\Gamma^1{}_{\beta\delta}+\Gamma^\beta{}_{1\delta}\ \Gamma^1{}_{\beta\lambda})A^{11}A^{11}\\
&\quad+(R^2{}_{1\delta\lambda}-\Gamma^\beta{}_{1\lambda}\ \Gamma^2{}_{\beta\delta}+\Gamma^\beta{}_{1\delta}\ \Gamma^2{}_{\beta\lambda})A^{21}A^{11}\\
&\quad+(R^3{}_{1\delta\lambda}-\Gamma^\beta{}_{1\lambda}\ \Gamma^3{}_{\beta\delta}+\Gamma^\beta{}_{1\delta}\ \Gamma^3{}_{\beta\lambda})A^{31}A^{11}\\
&\quad+\ \Gamma^2{}_{1\lambda}(A^{11}A^{21}{}_{,\delta}-A^{21}A^{11}{}_{,\delta})-\Gamma^2{}_{1\delta}(A^{11}A^{21}{}_{,\lambda}-A^{21}A^{11}{}_{,\lambda})\\
&\quad+\Gamma^3{}_{1\lambda}(A^{11}A^{31}{}_{,\delta}-A^{31}A^{11}{}_{,\delta})-\Gamma^3{}_{1\delta}(A^{11}A^{31}{}_{,\lambda}-A^{31}A^{11}{}_{,\lambda})=0. \qquad (33)
\end{aligned}
$$

The *RAA* terms in (33) group together to give

$$(R^1{}_{1\delta\lambda}\ A^{11}+R^2{}_{1\delta\lambda}\ A^{21}+R^3{}_{1\delta\lambda}\ A^{31})A^{11}$$

which either vanishes identically or vanishes by using equation (27). Calculating the derivatives of A's and re-expressing it by using the formula

$$R^\alpha{}_{\beta\gamma\delta;\lambda}=R^\alpha{}_{\beta\gamma\delta,\lambda}+\Gamma^\alpha{}_{\tau\lambda}\ R^\tau{}_{\beta\gamma\delta}-\Gamma^\tau{}_{\beta\lambda}\ R^\alpha{}_{\tau\gamma\delta}-\Gamma^\tau{}_{\gamma\lambda}\ R^\alpha{}_{\beta\tau\delta}-\Gamma^\tau{}_{\delta\lambda}\ R^\alpha{}_{\beta\gamma\tau}, \qquad (34)$$

we have

$$
\begin{aligned}
A''{}_{,\lambda}=&A^{11}{}_{;\lambda}+4\Gamma^1{}_{1\lambda}\ A^{11}+\Gamma^2{}_{1\lambda}(A^{12}+A^{21})+\Gamma^3{}_{1\lambda}(A^{13}+A^{31})\\
&+\Gamma^2{}_{1\lambda}\left(\begin{vmatrix}R^2{}_{231} & R^3{}_{231}\\ R^2{}_{112} & R^3{}_{112}\end{vmatrix}+\begin{vmatrix}R^2{}_{131} & R^3{}_{131}\\ R^2{}_{212} & R^3{}_{212}\end{vmatrix}\right)\\
&+\Gamma^3{}_{1\lambda}\left(\begin{vmatrix}R^2{}_{331} & R^3{}_{331}\\ R^2{}_{112} & R^3{}_{112}\end{vmatrix}+\begin{vmatrix}R^2{}_{131} & R^3{}_{131}\\ R^2{}_{312} & R^3{}_{312}\end{vmatrix}\right), \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
A^{21}{}_{,\lambda}=&A^{21}{}_{;\lambda}+4\Gamma^1{}_{1\lambda}\,A^{21}-\Gamma^1{}_{1\lambda}\,A^{21}+\Gamma^1{}_{2\lambda}\,A^{11}+\Gamma^3{}_{2\lambda}\,A^{31}+\Gamma^2{}_{1\lambda}\,A^{22}\\
&+\Gamma^3{}_{1\lambda}\,A^{23}+\Gamma^2{}_{2\lambda}\,A^{21}\\
&+\Gamma^2{}_{1\lambda}\left(\begin{vmatrix}R^3{}_{231} & R^1{}_{231}\\ R^3{}_{112} & R^1{}_{112}\end{vmatrix}+\begin{vmatrix}R^3{}_{131} & R^1{}_{131}\\ R^3{}_{212} & R^1{}_{212}\end{vmatrix}\right)\\
&+\Gamma^3{}_{1\lambda}\left(\begin{vmatrix}R^3{}_{331} & R^1{}_{331}\\ R^3{}_{112} & R^1{}_{112}\end{vmatrix}+\begin{vmatrix}R^3{}_{131} & R^1{}_{131}\\ R^3{}_{312} & R^1{}_{312}\end{vmatrix}\right). \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
A^{31}{}_{,\lambda}=&A^{31}{}_{;\lambda}+4\Gamma^1{}_{1\lambda}\,A^{31}-\Gamma^1{}_{1\lambda}\,A^{31}+\Gamma^2{}_{1\lambda}\,A^{32}+\Gamma^3{}_{1\lambda}\,A^{33}+\Gamma^1{}_{3\lambda}\,A^{11}\\
&+\Gamma^2{}_{3\lambda}\,A^{21}+\Gamma^3{}_{3\lambda}\,A^{31}\\
&+\Gamma^2{}_{1\lambda}\left(\begin{vmatrix}R^1{}_{231} & R^2{}_{231}\\ R^1{}_{112} & R^2{}_{112}\end{vmatrix}+\begin{vmatrix}R^1{}_{131} & R^2{}_{131}\\ R^1{}_{212} & R^2{}_{212}\end{vmatrix}\right)\\
&+\Gamma^3{}_{1\lambda}\left(\begin{vmatrix}R^1{}_{331} & R^2{}_{331}\\ R^1{}_{112} & R^2{}_{112}\end{vmatrix}+\begin{vmatrix}R^1{}_{131} & R^2{}_{131}\\ R^1{}_{312} & R^2{}_{312}\end{vmatrix}\right), \qquad (37)
\end{aligned}
$$

where

$$A^{11}{}_{;\lambda}\equiv(R^2{}_{131;\lambda}\,R^3{}_{112}-R^3{}_{131;\lambda}\,R^2{}_{112})+(R^2{}_{131}\,R^3{}_{112;\lambda}-R^3{}_{131}\,R^2{}_{112;\lambda}),$$

etc.. Substituting (35)-(37) into (33) and noticing the vanishing of the expression between (33) and (34), we arrive at

$$
\begin{aligned}
&\Gamma^2{}_{1\lambda}(A^{11}A^{21}{}_{;\delta}-\ A^{21}\ A^{11}{}_{;\delta})-\Gamma^2{}_{1\delta}(A^{11}\ A^{21}{}_{;\lambda}-A^{21}\ A^{11}{}_{;\lambda})+\Gamma^3{}_{1\lambda}(A^{11}\ A^{31}{}_{;\delta}-A^{31}\ A^{11}{}_{;\delta})\\
&\quad-\Gamma^3{}_{1\delta}(A^{11}\ A^{31}{}_{;\lambda}-A^{31}\ A^{11}{}_{;\lambda})+(-\ \Gamma^2{}_{1\delta}\ \Gamma^3{}_{1\lambda}+\Gamma^2{}_{1\lambda}\ \Gamma^3{}_{1\delta})B=0. \qquad (38)
\end{aligned}
$$

where

$$
\begin{aligned}
B=&\begin{vmatrix}R^2{}_{131} & R^3{}_{131}\\ R^2{}_{112} & R^3{}_{112}\end{vmatrix}\cdot\left(\begin{vmatrix}R^3{}_{331} & R^1{}_{331}\\ R^3{}_{112} & R^1{}_{112}\end{vmatrix}+\begin{vmatrix}R^3{}_{131} & R^1{}_{131}\\ R^3{}_{312} & R^1{}_{312}\end{vmatrix}\right)\\
&-\begin{vmatrix}R^3{}_{131} & R^1{}_{131}\\ R^3{}_{112} & R^1{}_{112}\end{vmatrix}\left(\begin{vmatrix}R^2{}_{331} & R^3{}_{331}\\ R^2{}_{112} & R^3{}_{112}\end{vmatrix}+\begin{vmatrix}R^2{}_{131} & R^3{}_{131}\\ R^2{}_{312} & R^2{}_{312}\end{vmatrix}\right)\\
&-\begin{vmatrix}R^2{}_{131} & R^3{}_{131}\\ R^2{}_{112} & R^3{}_{112}\end{vmatrix}\cdot\left(\begin{vmatrix}R^1{}_{231} & R^2{}_{231}\\ R^1{}_{112} & R^2{}_{112}\end{vmatrix}\ t\ \begin{vmatrix}R^1{}_{131} & R^2{}_{131}\\ R^1{}_{212} & R^2{}_{212}\end{vmatrix}\right)
\end{aligned}
$$

$$+\begin{vmatrix} R^1{}_{131} & R^2{}_{131} \\ R^1{}_{112} & R^2{}_{112} \end{vmatrix} \cdot \left(\begin{vmatrix} R^2{}_{231} & R^3{}_{231} \\ R^2{}_{112} & R^3{}_{112} \end{vmatrix} + \begin{vmatrix} R^2{}_{131} & R^3{}_{131} \\ R^2{}_{212} & R^3{}_{212} \end{vmatrix} \right) \tag{39}$$

In order for (38) to hold in every coordinates system the following must hold:

$$(A^{11} A^{21}{}_{;\lambda} - A^{21} A^{11}{}_{;\lambda}) = 0, \tag{40}$$

$$(A^{11} A^{31}{}_{;\lambda} - A^{31} A^{11}{}_{;\lambda}) = 0, \tag{41}$$

$$B = 0. \tag{42}$$

Conversely, if (40)-(42) hold, then (38) and (33) hold, and g_{11} can be integrated from (20). Note that the condition $B=0$ does not change under $2 \leftrightarrow 3$ in indices.

Because of permutation symmetry, to obtaion integrability conditions of equations (21) or (22) we simply permute the indices $1 \leftrightarrow 2$, or $1 \leftrightarrow 3$ respectively in equations (27), (40), (41) and (42) when written out in terms of the Riemannian tensor.

From the permutation $1 \leftrightarrow 2$ of (27) and following the method of deriving (14), (15), we get

$$g_{12} = -g_{22} \frac{R^2{}_{221} R^3{}_{223} - R^3{}_{221} R^2{}_{223}}{R^1{}_{221} R^3{}_{223} - R^1{}_{223} R^3{}_{221}} \,. \tag{43}$$

$$g_{23} = -g_{22} \frac{R^2{}_{223} R^1{}_{221} - R^2{}_{221} R^1{}_{223}}{R^1{}_{221} R^3{}_{223} - R^1{}_{223} R^3{}_{221}} \,. \tag{44}$$

Now we want to obtain conditions for (23) to be compatible with (28), (29), (43) and (44). From (43) and (44), we have

$$g_{23} = g_{12} \frac{R^2{}_{223} R^1{}_{221} - R^2{}_{221} R^1{}_{223}}{R^2{}_{221} R^3{}_{223} - R^2{}_{223} R^3{}_{221}} \,. \tag{45}$$

We express g_{12}, g_{13}, g_{22} and g_{23} in term of g_{11} from (28), (29), (43) and (45), and substitute into (23). After using the formula (34) and the conditions (40) and (41) to reduce, we obtain

$$C\Gamma^2{}_{1\delta} + D\Gamma^3{}_{1\delta} = 0, \tag{46}$$

where

$$\begin{aligned} C = & -(R^1{}_{221} R^3{}_{223} - R^1{}_{223} R^3{}_{221})(R^1{}_{112} R^3{}_{113} - R^1{}_{113} R^3{}_{112})(R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112}) \\ & + (R^2{}_{221} R^3{}_{223} - R^2{}_{223} R^3{}_{221})(R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112})(-R^3{}_{113} R^1{}_{212} - R^1{}_{112} R^3{}_{213} \\ & + R^1{}_{112} R^3{}_{132} + R^3{}_{112} R^1{}_{213} - R^3{}_{112} R^1{}_{132} + R^1{}_{113} R^3{}_{212}) + (R^2{}_{221} R^3{}_{223} - R^2{}_{223} R^3{}_{221}) \\ & \cdot (R^1{}_{112} R^3{}_{113} - R^1{}_{113} R^3{}_{112})(-R^3{}_{112} R^1{}_{113} - R^3{}_{112} R^2{}_{213} + R^3{}_{112} R^2{}_{132} - R^2{}_{113} R^3{}_{212} \\ & + R^3{}_{113} R^1{}_{112} + R^3{}_{113} R^2{}_{212} + R^2{}_{112} R^3{}_{213} - R^2{}_{112} R^3{}_{132}), \end{aligned} \tag{47}$$

$$\begin{aligned} D = & (R^2{}_{223} R^1{}_{221} - R^2{}_{221} R^1{}_{223})(R^1{}_{112} R^3{}_{113} - R^1{}_{113} R^3{}_{112})(R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112}) \\ & + (R^2{}_{221} R^3{}_{223} - R^2{}_{223} R^3{}_{221})(R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112})(-R^3{}_{113} R^1{}_{312} - R^3{}_{113} R^1{}_{132} \\ & - R^1{}_{112} R^3{}_{313} + R^3{}_{112} R^1{}_{313} + R^1{}_{113} R^3{}_{312} + R^1{}_{113} R^3{}_{132}) + (R^2{}_{221} R^3{}_{223} - R^2{}_{223} R^3{}_{221}) \\ & \cdot (R^1{}_{112} R^3{}_{113} - R^1{}_{113} R^3{}_{112})(-R^3{}_{112} R^2{}_{313} - R^2{}_{113} R^1{}_{112} - R^2{}_{113} R^3{}_{312} - R^2{}_{113} R^3{}_{132} \\ & + R^3{}_{113} R^2{}_{312} + R^3{}_{113} R^2{}_{132} + R^2{}_{112} R^1{}_{113} + R^2{}_{112} R^3{}_{313}). \end{aligned} \tag{48}$$

Equation (46) holds in every coordinate system, if and only if

$$C = D = 0 \tag{49}$$

If we express all g's in terms of g_{22} and substitute into (23), we obtain

$$C(1 \leftrightarrow 2) = D(1 \leftrightarrow 2) = 0. \tag{50}$$

Both (28) and (43) express g_{12} in terms of solutions of (20)-(21). Therefore the solutions of (20)-(21) must be compatible, i. e.,

$$g_{11} \frac{R^1{}_{112} R^3{}_{113} - R^1{}_{113} R^3{}_{112}}{R^2{}_{112} R^3{}_{113} - R^2{}_{113} R^3{}_{112}} = g_{22} \frac{R^2{}_{221} R^3{}_{223} - R^2{}_{223} R^3{}_{221}}{R^1{}_{221} R^3{}_{223} - R^1{}_{223} R^3{}_{221}} \,. \tag{51}$$

A calculation shows that the conditions for the derivatives of both sides of (51) to be equal are just

(49) and (50). Hence (49) and (50) guarantee that compatible solutions exist. Rut because of (51), the integration constants of g,, and g_{22} are related. There is only one independent integration constant.

Similarly, from equations (24) and (25), we obtain eight more conditions all from permutations of C and **D**.

Multiplying together Eq. (51) and the following two equations obtained from permutations of (51)

$$g_{22}\frac{R^2{}_{223}R^1{}_{221}-R^2{}_{221}R^1{}_{223}}{R^3{}_{223}R^1{}_{221}-R^3{}_{221}R^1{}_{223}}=g_{33}\frac{R^3{}_{332}R^1{}_{331}-R^3{}_{331}R^1{}_{332}}{R^2{}_{332}R^1{}_{331}-R^2{}_{331}R^1{}_{332}}, \tag{52}$$

$$g_{33}\frac{R^3{}_{331}R^2{}_{332}-R^3{}_{332}R^2{}_{331}}{R^1{}_{331}R^2{}_{332}-R^1{}_{332}R^2{}_{331}}=g_{11}\frac{R^1{}_{113}R^2{}_{112}-R^1{}_{112}R^2{}_{113}}{R^3{}_{113}R^2{}_{112}-R^3{}_{112}R^2{}_{113}}, \tag{53}$$

we have the following condition

$$(R^1{}_{112}R^3{}_{113}-R^1{}_{113}R^3{}_{112})(R^2{}_{223}R^1{}_{221}-R^2{}_{221}R^1{}_{223})(R^3{}_{331}R^2{}_{332}-R^3{}_{332}R^2{}_{331})$$
$$=(R^2{}_{221}R^3{}_{223}-R^2{}_{223}R^3{}_{221})(R^3{}_{332}R^1{}_{331}-R^3{}_{331}R^1{}_{332})(R^1{}_{113}R^2{}_{112}-R^1{}_{112}R^2{}_{113}). \tag{54}$$

for the cyclic consistency of the solutions.

Summarizing the results in this section, we obtain the following theorem:

THEOREM: The necessary and sufficient conditions, in general, for the existence of a metric locally in three-dimensional affine manifolds are the following thirty-seven equations:

i) Eighteen involve covariant derivatives of $R^\alpha{}_{\beta\gamma\delta}$ as

$$(R^2{}_{131}R^3{}_{112}-R^3{}_{131}R^2{}_{112})(R^1{}_{131}R^3{}_{112}-R^3{}_{131}R^1{}_{112})_{;\lambda}$$
$$=(R^2{}_{131}R^3{}_{112}-R^3{}_{131}R^2{}_{112})_{;\lambda}(R^1{}_{131}R^3{}_{112}-R^3{}_{131}R^1{}_{112})$$

and their different permutations, and

ii) The remaining nineteen not involving covariant derivatives of $R^\alpha{}_{\beta\gamma\delta}$ are (26) and its two different permutations, $B=0$ and its two different permutations, $C=D=0$ and their ten different permutations and (54).

In case the above conditions are satisfied, the solutions are

$$\begin{aligned}g_{11}=\exp\Big\{2\int_{(x_0,y_0,z_0)}^{(x,y,z)}&\Big(\Gamma^1{}_{11}-\Gamma^2{}_{11}\frac{R^1{}_{112}R^3{}_{113}-R^1{}_{113}R^3{}_{112}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{112}}\\&-\Gamma^3{}_{11}\frac{R^1{}_{113}R^2{}_{112}-R^1{}_{112}R^2{}_{113}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{113}}\Big)dx\\&+\Big(\Gamma^1{}_{12}-\Gamma^2{}_{12}\frac{R^1{}_{112}R^3{}_{113}-R^1{}_{113}R^3{}_{112}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{112}}\\&-\Gamma^3{}_{12}\frac{R^1{}_{113}R^2{}_{112}-R^1{}_{112}R^2{}_{113}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{112}}\Big)dy\\&+\Big(\Gamma^1{}_{13}-\Gamma^2{}_{13}\frac{R^1{}_{112}R^3{}_{113}-R^1{}_{113}R^3{}_{112}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{112}}\\&-\Gamma^3{}_{13}\frac{R^1{}_{113}R^2{}_{112}-R^1{}_{112}R^3{}_{113}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{112}}\Big)dz+C\Big\}\end{aligned}$$

$$g_{12}=-g_{11}\frac{R^1{}_{112}R^3{}_{113}-R^1{}_{113}R^3{}_{112}}{R^2{}_{112}R^3{}_{113}-R^2{}_{113}R^3{}_{112}}$$

and similar expressions for other g ' s.

The integration constants in the expressions of g_{22} and g_{33} are related to C, through (51) and the like. Hence the only freedom in the solution is the constant scale factor. This also proves that spaces of essentially different Riemannian signatures cannot have the same connection.